Power Round

Names: ______________________________

Team Name: ______________________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 90 minutes to solve the problems during this round.
3. Your submission will be graded and assigned point values out of the total points possible per problem. Your total score will be the sum of the points you receive for each problem.
4. Submissions will be graded on correctness as well as clarity of proof. A proof with significant progress towards a solution may receive more credit than a correct answer with no justification.
5. **You may use the result of a previous problem in the proof of a later problem, even if you do not submit a correct solution to the referenced problem.** However, you may not use the result of a later problem in the proof of an earlier problem.
6. Please submit each part of each problem on a separate page. Write your team name, problem number, and page number clearly at the top of each page.
7. No calculators or electronic devices are allowed.
8. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
9. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
10. Do not discuss the problems with anyone outside of your team until all papers have been collected.
11. If you have a question or need to leave the room for any reason, please raise your hand quietly.
12. Good luck!

Acceptable Answers

1. Solutions should be written in proof format. All answers, reasoning, and deductions must be explained and justified, unless the problem explicitly asks for you to “compute”. Problems asking you to “show”, “prove”, or “justify” **require proof!**
2. Proofs will be graded both on correctness as well as clarity of presentation.
3. Partial credit may be awarded for significant progress towards a solution.
4. Each problem must be written starting on a new, blank page. Two different problems should not be written on the same page.
5. At the top right corner of each page, please clearly print your team name, problem number, and page number.
6. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Progression Predicament

Sequences are an essential tool in mathematics that allow us to make statements about large (even infinite!) sets of numbers and their properties. One of the first types of sequences we learn about is the arithmetic sequence, where each pair of successive terms has the same difference. But, what if we were to construct a sequence that restricted the presence of such sequences?

Notation: Let $\mathbb{Z}$ denote the integers, $\mathbb{N}$ denote the positive integers $1, 2, 3, \ldots,$, and $\mathbb{Z}_{\geq 0}$ denote the nonnegative integers $0, 1, 2, \ldots$. $\{a_i\}_{i=1}^n$ denotes a sequence $a_1, a_2, \ldots, a_n$, while $\{a_i\}_{i=1}^\infty$ denotes an infinite sequence $a_1, a_2, a_3, \ldots$.

We say that an arithmetic sequence is in a set of integers S if there exist $a, b, c \in S$ such that $b - a = c - b$. If S is a sequence, a, b, c need not be consecutive terms. Make a definition of geometric sequences being in S the same way. For example, the finite sequence $1, 2, 4, 5, 6$ contains the arithmetic sequence $2, 4, 6$.

For convenience, define $A(S)$ to be the length of the longest arithmetic sequence in set S, and we define $G(S)$ to be the length of the longest geometric sequence in set S.

Problem 1. Let the first two terms of sequence $\{a_n\}_{n=0}^\infty$ be $a_0 = 0$ and $a_1 = 1$. Let subsequent terms be chosen such that for all $k \geq 2$, a_k is the smallest integer greater than a_{k-1} such that $A(a_0, a_1, \ldots, a_{k-1}, a_k) < 3$, i.e. $a_k = \min\left(\{n \in \mathbb{Z} \mid n > a_{k-1} \text{ and } A\left(\{a_i\}_{i=0}^{k-1} \cup \{n\}\right) < 3\}\right)$.

(a) (1 Point) List out the first ten terms of this sequence (up to a_9).

(b) (1 Point) The problem statement implicitly assumes that such a sequence has an infinite number of terms. Justify this assumption.

Solution:

(a) $0, 1, 3, 4, 9, 10, 12, 13, 27, 28$

(b) Suppose, by contradiction, that there is a last term a_N. The largest difference between two terms of $a_0, \ldots, a_N$ is $a_N - a_0 = a_N$ because the sequence is strictly increasing. Thus the largest integer eliminated by the arithmetic sequence condition is $2a_N$. If every integer in $[a_N + 1, 2a_N - 1]$ is eliminated by the arithmetic sequence condition, then $2a_N + 1$ is a term in the sequence, contradicting our assumption that there is a last term.

Problem 2. While it was relatively easy to calculate the first few terms of the sequence defined in Problem 1, finding subsequent terms by checking every possible arithmetic progression becomes an increasingly lengthy process. As such, we would like to find an explicit formula for the terms of $\{a_n\}_{n=0}^\infty$.

(a) (1 Point) List out the first ten terms in the sequence in base 3.

(b) (4 Points) Find an explicit expression (either numerical or descriptive), for the sequence given in Problem 1, and prove that it is valid.

(c) (1 Point) Compute the value of a_{100}.

Solution:

(a) $0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001$

(b) We claim that a_n is the binary representation of n interpreted in ternary (base 3). This is equivalent to a non-negative integer being in $\{a_n\}_{n=0}^{\infty}$ if and only if its ternary representation consists of only 0's and 1's. We proceed using induction. For 1-digit ternary numbers, we have 0 and 1 are terms in $\{a_n\}_{n=0}^{\infty}$ but $0, 1, 2$ is an arithmetic sequence so 2 is not a term in the sequence. Assume that all ternary numbers with k or fewer digits consisting of only 0's and 1's are terms in the sequence and that these are the only terms. The greatest of these numbers is $\frac{3^k-1}{2}$ and so the largest number that can form an arithmetic sequence with two numbers with k digits or less is $3^k - 1$. Because of this, we know 3^k is a term in the sequence. The first number that forms an arithmetic sequence with one of the $k+1$ digit terms and one of the k-or-fewer digit terms is $3^k + (3^k - \frac{3^k-1}{2}) = \underbrace{1\ldots1}_{k-1\text{ 1's}}2$. Thus we can simply add 3^k to all of the terms with k or fewer digits to get all of the terms with $k+1$ digits. Now, we need only show that any $k+1$-digit number with a 2 in its ternary representation would be a part of an arithmetic sequence. Consider two functions $f_{2\to1}$ and $f_{2\to0}$ that take the digits of a ternary number and replaces all of the 2's with 1's and 0's, respectively. For a ternary number n with at least one digit being 2, the following is an arithmetic sequence: $f_{2\to0}(n), f_{2\to1}(n), n$ so no $k+1$-digit numbers can have a 2 in their ternary representation and be in the sequence. By induction on k, we have shown that the elements of the sequence are exactly the numbers whose ternary representations contain only 0's and 1's.

(c) $100 = 1100100_2 \implies a_{100} = 1100100_3 = \boxed{981}$

Problem 3. Now, instead of seeding our sequence with $a_0 = 0$ and $a_1 = 1$, we seed it with $a_0 = 0$ and $a_1 = 3^k$ for some $k \in \mathbb{Z}$ such that $k \geq 1$. For example, if we choose $k = 2$, the first four terms of the sequence are $0, 9, 10, 12$.

(a) (6 Points) Find an explicit expression (either numerical or descriptive), for the n^{th} term of $\{a_n\}_{n=0}^{\infty}$, and prove that it is valid.

(b) (2 Points) Compute the value of a_{50} when $k = 4$.

Solution:

(a) The following is one characterization of the sequence. Let $a(k, n)$ be the n^{th} term of the sequence starting with $0, 3^k$. We can then express $a(k, n)$ as follows where $m \in \mathbb{Z}_{\geq 0}$:

$$a(k,n) = \begin{cases} a(0,n), & \text{if } n = 2^{k+1}m \\ 3^k + a(0, n-1), & \text{if } n \in \{2^{k+1}m + 1, \ldots, 2^{k+1}m + 2^k\} \\ 3^k + a(0,n), & \text{if } n \in \{2^{k+1}m + 2^k + 1, \ldots, 2^{k+1}m + 2^{k+1} - 1\} \end{cases}$$

We start by showing this is true for $m = 0$:

Clearly, $a(k, 0) = 0$. We can then copy the terms of $a(0, n)$ that are less than 3^k and add 3^k to each of them to get the next 2^k terms of the sequence. We know these are the correct terms in our new sequence because shifting all of the terms in an arithmetic sequence-free set by the same number maintains this property. We stop before 3^k because then we would have $0, 3^k, 2 \cdot 3^k$ in our new sequence. Because we had an initial term and then added on the shifted copy, we subtract 1 from the second index of $a(0, n)$ to get the next part of our formula.

Because we leave out the 3^k term from $a(0, n)$ in $a(k, n)$. We shift the second index of $a(0, n-1)$ forward by 1 to get $a(k, n) = 3^k + a(0, n)$ for the next group of terms. Using the functions constructed in the proof of 2(b), note that the first term not part of $a(0, n)$ that has 3^k in its sequence is $2 \cdot 3^k$, so $2 \cdot 3^k + 3^k = 3^{k+1}$ is a term in $a(k,n)$, and this is the 2^{k+1}-th term. Thus we have shown the formula holds for $m = 0$.

From the formula for $a(k,n)$, it is not to hard to see that terms of our desired sequence are of the form $\underbrace{\#\ldots\#}_{\text{Block 1}}\#\underbrace{\#\ldots\#}_{\text{Block 2}}$, such that the following are all true:

- the middle digit not in either block is the $k+1$-th digit from the right
- each digit of Block 1 is 0 or 1
- the middle digit is 0 or 1 or 2
- if the middle digit is 0, each digit of Block 2 is 0
- if the middle digit is 1, each digit of Block 2 is either 0 or 1
- if the middle digit is 2, each digit of Block 2 is either 0 or 1 with not all digits being 0

By assuming our formula holds up to an arbitrary $m \geq 0$, we are implicitly assuming this digit characterization holds for terms with at most $m+2$ digits.

Now, consider a number A with $m+3$ digits, that we will write as B_1DB_2 where B_1 and B_2 are blocks as described above and D is the middle digit. We further assume the above digit characterization of terms in the sequence holds for all numbers less than A.

First we show that if A fits the digit characterization that there is no arithmetic sequence ending with A. Suppose there is such a sequence $A'', A', A = B_1''D''B_2'', B_1'D'B_2', B_1DB_2$.

If $D = 0$, then $B_2 = 0\ldots0$, and we are left with B_1 followed by all 0's. We know ternary numbers consisting of only 0's and 1's do not form arithmetic progressions, therefore $D' = 2$. For such a sequence to hold, we must have $B_2' = 0\ldots0$ and $B_2'' = 0\ldots0$, otherwise there would be a nonzero digit in B_2. To have $D = 0$, we must then have $D' = 2$ and $D'' = 1$, but then $B_1'D'B_2'$ does not fit the stated digit characterization and thus there is no such sequence.

Now, consider $D = 1$ or $D = 2$. Working from the rightmost digit, it is clear to see we must have $B_2 = B_2' = B_2''$. Thus, $D = D' = D''$ (note that $D = 2, D' = 1, D'' = 0$ is not possible as this would force $B_2 = 0\ldots0$ or $B_2'' \neq 0\ldots0$), and there is no carry over into the B_1's. By similar logic as before, we must also have $B_1 = B_1' = B_1''$, but then we have $A = A' = A''$. Therefore, there is no such sequence.

Now, suppose A does not fit the digit characterization above.

Suppose $D = 0$. If $B_2 = 0\ldots0$, then set $B_1' = f_{2\to1}(B_1), B_1'' = f_{2\to0}(B_1), D' = D'' = 0, B_2' = B_2'' = 0\ldots0$. Otherwise, set $B_2' = f_{2\to1}(B_2), B_2'' = f_{2\to0}(B_2), D' = 2, D'' = 1$. Define $g_{2\to1}(x)$ to match each digit given by $f_{2\to1}(x)$ except the last digit becomes 0 if the last digit of x is 1, and it becomes 1 otherwise. Similarly, define $g_{2\to0}(x)$ such that it matches each digit given by $f_{2\to0}(x)$ except the last digit becomes 1 if the last digit of x is 2, and it becomes 0 otherwise. Set $B_1' = g_{2\to1}(B_1)$ and $B_1'' = g_{2\to0}(B_1)$. This ensures that A'', A', A form a valid arithmetic progression (i.e. not all equal).

Suppose $D = 1$ or $D = 2$. Then, we set $B_1' = f_{2\to1}(B_1), B_1'' = f_{2\to0}(B_1), D' = D'' = D, B_2' = f_{2\to1}(B_2), B_2'' = f_{2\to0}(B_2)$, unless $D = 2$ and B_2 consists only of 2's. In this case, set $B_1' = f_{2\to1}(B_1), B_1'' = f_{2\to0}(B_1), D' = 1, D'' = 2, B_2' = f_{2\to1}(B_2), B_2'' = f_{2\to0}(B_2) = 0\ldots0$. This ensures that A'', A', A form a valid arithmetic progression (i.e. not all equal).

Therefore, we have shown that the digit characterization derived from our formula holds. By induction on the value of A, this holds for all terms of the sequence, and thus our given formula holds.

(b) $50 \in \{2^{4+1} \cdot 1 + 2^4 + 1, \ldots, 2^{4+1} \cdot 1 + 2^{4+1} - 1\} \implies a(4,50) = 3^4 + a(0,50) = 3^4 + 327 = \boxed{408}$.

Problem 4. This time, we seed the sequence with $a_0 = j$ and $a_1 = j + 3^k$ for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.

(a) (2 Points) Find an explicit expression (either numerical or descriptive), for the n^{th} term of $\{a_n\}_{n=0}^{\infty}$, and prove that it is valid.

(b) (1 Point) Compute the value of a_{50} when $j = 6$ and $k = 2$.

Solution:

(a) Because we are only concerned with the differences between terms, we can simply shift the formula found in 3(a) by j. Let $a(j, k, n)$ be the n^{th} term of the sequence starting with $j, j + 3^k$. We can then express $a(j, k, n)$ as follows where $m \in \mathbb{Z}_{\geq 0}$:

$$a(j,k,n) = \begin{cases} j + a(0,0,n), & \text{if } n = 2^{k+1}m \\ j + 3^k + a(0,0,n-1), & \text{if } n \in \{2^{k+1}m + 1, \dots, 2^{k+1}m + 2^k\} \\ j + 3^k + a(0,0,n), & \text{if } n \in \{2^{k+1}m + 2^k + 1, \dots, 2^{k+1}(m+1) - 1\} \end{cases}$$

(b) $50 \in \{2^{2+1} \cdot 6 + 1, \dots, 2^{2+1} \cdot 6 + 2^2\} \to a(6, 2, 50) = 6 + 3^2 + a(0, 0, 49) = \boxed{340}$.

Problem 5. Now, we consider sequences that allow for arithmetic sequences of certain lengths but forbid all longer ones. Let $M \in \mathbb{Z}$ such that $M \geq 3$. Let the first $M - 1$ terms of sequence $\{a_n\}_{n=0}^{\infty}$ be defined by $a_n = n$ for all $n \in \{0, \dots, M - 2\}$. Let subsequent terms be chosen such that for all $k \geq M - 1$, a_k is the smallest integer greater than a_{k-1} such that there does not exist an arithmetic sequence of length greater than or equal to M in the set $\{a_0, \dots, a_k\}$, i.e. $a_k = \min\left(\{n \in \mathbb{Z} \mid n > a_{k-1} \text{ and } A\left(\{a_i\}_{i=0}^{k-1} \cup \{n\}\right) < M\}\right)$.

(a) (5 Points) Suppose M is prime. Find an explicit expression (either numerical or descriptive), for the n^{th} term of $\{a_n\}_{n=0}^{\infty}$, and prove that it is valid.

(b) (1 Point) Compute the value of a_{2024} for $M = 11$.

Solution:

(a) This uses essentially the same proof as 2(b), but replace usage of base 3 with base p. All details of the proof can be adjusted to work with this arbitrary (but still prime) base. a_n can be obtained by interpreting n as a base p number and converting back to base 10.

(b) $2024 = 2024_{10} \implies a_{2024} = 2024_{11} = \boxed{2688}$.

Problem 6. Now, consider disallowing geometric sequences instead. Let the first two terms of sequence $\{g_n\}_{n=0}^{\infty}$ be $g_0 = 1$ and $g_1 = 2$. Let subsequent terms be chosen such that for all $k \geq 2$, g_k is the smallest integer greater than g_{k-1} such that $G\left(\{g_n\}_{n=0}^{k}\right) < 3$, i.e. $g_k = \min\left(\{n \in \mathbb{Z} \mid n > g_{k-1} \text{ and } G\left(\{g_i\}_{i=0}^{k-1} \cup \{n\}\right) < 3\}\right)$.

(a) (1 Point) List out the first 20 terms of this sequence (up to g_{19}).

(b) (2 Points) The problem statement implicitly assumes that such a sequence has an infinite number of terms. Justify this assumption.

Solution:

(a) $1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 24, 26$

(b) Suppose for contradiction that this sequence has a largest element r. Notice that no geometric progressions can be made by adding r^2+1, since the largest ratio between terms is r and the largest progression that can be made is $1, r, r^2$. Thus there exists a larger element that can be added, contradicting the maximality of the existing sequence.

Problem 7. As with the arithmetic progression-free sequences, we would like to have a nicer characterization for the terms of sequence $\{g_n\}_{n=0}^{\infty}$.

(a) (1 Point) Write out the prime factorizations of the first 20 terms of this sequence. What do you notice about the exponents on the prime numbers?

(b) (6 Points) Find an explicit expression (either numerical or descriptive) for elements in the sequence $\{g_n\}_{n=0}^{\infty}$, and prove it is valid.

Solution:

(a) $1, 2, 3, 5, 2 \cdot 3, 7, 2^3, 2 \cdot 5, 11, 13, 2 \cdot 7, 3 \cdot 5, 2^4, 17, 19, 3 \cdot 7, 2 \cdot 11, 23, 2^3 \cdot 3, 2 \cdot 13.$

(b) Let $\mathcal{G} = \{g_n : n \in \mathbb{N}\}$. I claim that $\mathcal{G}$ contains all positive integers whose prime exponents are elements of the minimal AP-avoiding sequence in problem 1. Firstly, 1 and 2 are the smallest positive integers that satisfy this. Now, suppose $g_1, g_2, \ldots, g_n$ are chosen greedily in the way described. Note that it contains all primes in between g_1 and g_n. When considering g_{n+1}, the next greatest element following the stated rules clearly don't form a geometric sequence with any previous terms, since a geometric sequence is made if and only if the exponents on in prime factorizations of adjacent terms make an arithmetic progression. No smaller terms can be taken either, since otherwise each exponent of a prime factor that isn't in the AP-avoiding sequence will form an arithmetic sequence with previous terms by construction. The claim follows.